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Results in Mathematics

On the shape operator of surfaces in space forms

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Dedicated to S. S. Chern on the occasion of his 90^{th} birthday

Abstract

Necessary and sufficient conditions are studied for the existence of surfaces in space forms with prescribed shape operator S. We consider three cases: that of the non-existence of a solution, the case of a unique solution and that of more than one solution. Examples are given for local and global S-deformations and S-rigidity.

Keywords: Surfaces in space forms, prescribed shape operator.

MSC 2000: 53 A 05, 53 C 42

1 Introduction

Recently U. Simon has raised the question whether two ovaloids in Euclidian space $\mathbb{R}^3$, having the same shape operator (Weingarten operator) S in corresponding points, are congruent. In [3] this rigidity result is proved under the additional assumption that the mapping between the two surfaces is area preserving. This additional assumption is not needed when the surfaces are of genus zero and one of the surfaces is a surface of revolution [9].
In the special case of surfaces of revolution, the corresponding local problem has been completely discussed in [9].
In this paper, I deal with the question whether an arbitrary diagonalizable endomorphism field S on a 2-manifold M can appear - at least locally - as shape operator of a surface, that means, whether there is an immersion of M into a Riemannian 3-space $(\widetilde{M}, \widetilde{g})$ which induces S.
For hypersurfaces in arbitrary Riemannian spaces $\widetilde{M}$, there are no restrictions on S, as is pointed out in section 2.
In the case of surfaces in 3-dimensional space forms $\widetilde{M}(c)$ (constant sectional curvature c), which is studied here, an immersion of M into $\widetilde{M}(c)$ is determined - up to motions in $\widetilde{M}$ -

by S and the induced metric g on M, such that the equations of Gauss and Codazzi yield conditions to find g, if S is prescribed on M. Since S has to be self-adjoint with respect to g, the PDE-problem is overdetermined.
If k_1, k_2 are arbitrary differentiable functions $M \to \mathbb{R}$ with different values and $rank(k_1, k_2) = \rho = 2$, then obviously k_1, k_2 appear as principal curvatures of surfaces in $\mathbb{R}^3$. But if $\rho \leq 1$, not all functions k_1, k_2 are possible, e. g. as a consequence of Weyl's identity. On the other hand there are restrictions on the eigendirections e_1, e_2 of S near an umbilic, following from Hamburgers's index theorem.
If S is prescribed, and if u, v are local coordinates such that the coordinate lines are the integral curves of e_1, e_2 (curvature lines), then k_1, k_2 are not at all arbitrary functions of u, v. There are three cases:

- **Case 1:** *There is no surface with shape operator S.*
- **Case 2:** *There is exactly one surface (up to motions).*
- **Case 3:** *There is a pair of non congruent surfaces with the same S.*

In case 3, the surfaces are called *S-deformable*; in case 2 the surface is called *S-rigid.*

Below, I will give conditions for S which are necessary and sufficient for the three cases. Some examples are discussed where the conditions can be evaluated. There are nontrivial cases of area-preserving S-deformability and such with global but not local S-rigidity.

A first version of my considerations has been presented during a workshop at the TU Berlin in December 2000. In the meantime I have learned that already in 1933, S. P. Finikoff and B. Gambier [5] have published a comprehensive investigation of S-deformable surfaces, which recently has been taken up and extended by E. V. Ferapontov [4]. In 1945, E. Cartan [2] has shown that the determination of all S-deformable surfaces depends on six functions of one variable. He also studied the case where the curvature lines are preserved, but the role of k_1, k_2 is interchanged, i. e. the second surface has the shape operator $(k_1 + k_2)\, id - S$.

In the previous papers, the authors work with the third fundamental form $III(X,Y) = g(S(X), S(Y))$; our conditions for the first fundamental form I include surfaces where zeros of $k_1 k_2$ are admitted; for surfaces with $k_1 k_2 \neq 0$, the conditions in terms of I and those in terms of III are equivalent.

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2 Hypersurfaces in Riemannian spaces

Let $\widetilde{M}$ be an (n+1)-dimensional space with metric $\widetilde{g}$ and Levi-Civita connection $\widetilde{\nabla}$, and M an oriented n-dimensional submanifold with induced metric g and connection ∇. If X, Y are tangent vector fields on M and N is the unit normal of M then

$$\widetilde{\nabla}_X Y = \nabla_X Y + b(X,Y)N; \qquad \widetilde{\nabla}_X N = -S(X),$$

where $b(X,Y) = g(S(X),Y)$ is the second fundamental form II. The integrability conditions are

$$(1)\quad \begin{aligned} R(X,Y,X,Y) &= \widetilde{R}(X,Y,X,Y) + b(X,X)b(Y,Y) - b(X,Y)^2 \\ (\nabla_X S)Y - (\nabla_Y S)X &= -\widetilde{R}(X,Y)N \end{aligned}$$

Proposition 1: *If g and S are arbitrarily prescribed on M such that S is g-self-adjoint and the eigenvalues of S are bounded, then there exists a manifold $\widetilde{M} \supset M$ and a metric $\widetilde{g}$ on $\widetilde{M}$ such that on M the given metric g and shape operator S coincide with the induced tensor fields.*

Proof: We take $\widetilde{M} = M \times I$, $I = \{t \mid -\varepsilon < t < \varepsilon\}$. For vector fields X, Y on $\widetilde{M}$, tangent to the leaves $M \times \{t\}$ and $N = \partial_t$, we define

$$\widetilde{g}(X,Y) = g(X,Y) - 2tg(S(X),Y);\;\; \widetilde{g}(X,N) = 0;\;\; \widetilde{g}(N,N) = 1.$$

With coordinates $u^2, \dots, u^{n+1}$ in M, we write for the Gauss basis in $\widetilde{M}$: $\partial_i = E_i$ $(i = 2, \dots, n+1)$ and $\partial_t = E_1$. Now

$$\begin{aligned} \widetilde{\nabla}_{E_i} N &= -S(E_i) = \widetilde{\nabla}_{E_i} E_1 = \widetilde{\Gamma}^j_{i1} E_j \\ \widetilde{\Gamma}^j_{i1} &= \widetilde{g}^{jk} \widetilde{\Gamma}_{i1,k} = \frac{1}{2} \widetilde{g}^{jk} \frac{\partial \widetilde{g}_{ik}}{\partial t}. \end{aligned}$$

For $t = 0$ this yields the assertion. ■

Example: *The standard sphere $M = S^2$ can be imbedded in $\widetilde{M}$ as minimal surface with one umbilic of index 2:*
Using complex coordinates $z = x_1 + ix_2$, the standard metric of S^2 is given by

$$ds^2 = \frac{4dzd\bar{z}}{(1+z\bar{z})^2}, \quad z \in \mathbb{C} \text{ and } z = \frac{1}{w} \text{ near } w = 0.$$

Let S be defined by the representation matrix

$$(S^i{}_j) = \frac{1}{(1+z\bar{z})^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

the matrix of II and the Hopf differential [7], p. 137, are

$$\begin{aligned} &\frac{4}{(1+z\bar{z})^4} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ \Phi dz^2 &= \frac{4}{(1+z\bar{z})^4} dz^2 = \frac{4\bar{w}^4}{(1+w\bar{w})^4} dw^2 \quad (z = 1/w). \end{aligned}$$

S is defined on all of M with a zero at $z = \infty$.

3 Surfaces in space forms

We take $n = 2$ and a 3-dimensional space form $\widetilde{M}(c)$, e.g. one of the standard spaces $\widetilde{M}(0) = \mathbb{R}^3$, $\widetilde{M}(1) = S^3$ or $\widetilde{M}(-1) = \mathbb{H}^3$. Now (1) takes the form

$$\begin{aligned} (2) \qquad K_I &= c + k_1 k_2, \quad (K_I = \frac{R_{1212}}{detg}) \\ (3) \qquad (\nabla_X S)(Y) &= (\nabla_Y S)(X). \end{aligned}$$

In local coordinates u, v the operator S has a representation matrix $(S^i{}_j) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, and (3) has the form

$$(4) \qquad \begin{aligned} A_v - B_u &= -(A-D)\Gamma^1_{12} + B\Gamma^1_{11} - C\Gamma^1_{22} \\ C_v - D_u &= -(A-D)\Gamma^2_{12} + B\Gamma^2_{11} - C\Gamma^2_{22}. \end{aligned}$$

If $k_1 \neq k_2$, S is determined by the eigenvalues k_1, k_2 and two linearly independent vector fields e_1, e_2 as eigendirections. We use the notation $2H = k_1 + k_2 = \operatorname{trace} S$, $K = k_1 k_2 = \det S$.

3.1 Conditions for k_1, k_2

If $k_1(u, v) \neq k_2(u, v)$ are arbitrary functions with $dk_1 \wedge dk_2 \neq 0$ in a neighbourhood of $u = v = 0$, then there exist surfaces with principal curvatures k_1, k_2.

Proof: Choose an arbitrary surface with parameters x, y and independent principal curvatures $k_1^*(x, y), k_2^*(x, y)$ which in $(0, 0)$ take the same values $a \neq b$ as k_1, k_2; e. g. in $\mathbb{R}^3$ the surface $(x, y, \frac{1}{2}(ax^2 + by^2) + \frac{1}{6}(x^3 + y^3))$ has this property. The equations $k_1^* = k_1$, $k_2^* = k_2$ define a local diffeomorphism $(x, y) \leftrightarrow (u, v)$; with respect to u, v the surface has the prescribed principal curvatures. ■

3.2 Conditions for e_1, e_2 near an umbilic

Example:

$$(S^i{}_j) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} u & v \\ v & -u \end{pmatrix}$$

is not the shape operator of a surface in $\widetilde{M}(c)$.

First Proof: The second equation (4) does not hold for $u = v = 0$. ■

Second Proof: The principal directions are given by

$$(A - D)dudv - Cdu^2 + Bdv^2 = 0$$

which in the case $B = C$ can be written in the form

$$Im(\Psi dw^2) = 0, \qquad w = u + iv, \qquad \Psi = \frac{1}{2}(A - D) - iB.$$

For $\Psi = \bar{w}$, the Poincaré index of the umbilic is $j = \frac{1}{2}$; but $H = 0$ implies $j < 0$. See H. Hopf [7], p. 139. ■

More general examples result if we take homogeneous polynomials in u, v of degree n: Since the right hand side of (4) vanishes of order n, the left hand side is identically zero; thus e. g. $\Psi = \bar{w}^n$ for $n \geq 3$ ($B = C$, $j = n/2$) is not possible on surfaces in $\widetilde{M}(c)$.

Remark: Hamburger has proved his index theorem (namely that $j \leq 1$, see [8]) also for non-analytic surfaces, if the Taylor expansion of $H^2 - K$ starts with a definite term ("regular case" in the paper of G. Bol [1]). Further restrictions near an umbilic appear in the case of Weingarten surfaces ($dk_1 \wedge dk_2 \equiv 0$) or if certain inequalities hold for k_1, k_2, see [6], [10], [11].

4 Weyl's identity

In a 2-manifold (M, g) the change of an orthonormal frame e_1, e_2 is described by

$$\begin{aligned} \nabla_X e_1 &= \omega(X) e_2, \\ \nabla_X e_2 &= -\omega(X) e_1 \end{aligned}$$

The integrability condition is

$$R(X, Y)e_1 = d\omega(X, Y)e_2,$$

where $d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])$. This means $d\omega(e_1, e_2) = -K_I$. Writing $\omega(X) = g(V, X)$ with $V = \omega(e_1)e_1 + \omega(e_2)e_2$ and passing from V to $W = V^\perp$, we get

(5) $$W = -\omega(e_2)e_1 + \omega(e_1)e_2 \Rightarrow div\, W = K_I.$$

Let e_1, e_2 be eigenvectors of a self-adjoint operator S with property (3). Differentiating $S(e_\beta) = k_\beta e_\beta$ and applying (3), one sees that (3) is equivalent to

(6) $$\omega(e_1) = \frac{e_2 k_1}{k_1 - k_2}, \quad \omega(e_2) = \frac{e_1 k_2}{k_1 - k_2}.$$

Inserting (6) into (5) and calculating $div\, W = g(\nabla_{e_1} W, e_1) + g(\nabla_{e_2} W, e_2)$ there results an expression in the derivatives of k_1, k_2. Using the Hesse operator $\mathcal{H}_f(X, Y) = XYf - \nabla_X Y f$ (second covariant differential of the function f), we conclude

Proposition 2 *(Weyl identity): For the shape operator of a surface of class C^4 in $\widetilde{M}(c)$ with eigenvalues $k_1 \neq k_2$ and orthonormal eigenvectors e_1, e_2, the following identity holds:*

(7) $$\frac{\mathcal{H}_{k_1}(e_2, e_2) - \mathcal{H}_{k_2}(e_1, e_1)}{k_1 - k_2} - 2\frac{(e_2 k_1)^2 + (e_1 k_2)^2}{(k_1 - k_2)^2} = c + k_1 k_2.$$

Examples of conclusions from (7) are

Proposition 3: *(a) If S is the shape operator of a surface in $\widetilde{M}(c)$, and if there is a point $p \in M$ such that $k_1(p) > k_2(p)$ and $k_1 = max$, $k_2 = min$ in p, then $k_1k_2 \leq -c$.*
(b) If $dk_1 = dk_2 = 0$ in p, then $\frac{e_2e_2k_1 - e_1e_1k_2}{k_1 - k_2} = c + k_1k_2$.
(c) For a surface in $\widetilde{M}(c)$ with constant $k_1 \neq k_2$ (isoparametric surface) there holds $k_1k_2 = -c$.

Corollary: *$k_1 = max$, $k_2 = min$ in p and $k_1(p) > k_2(p) > 0 \Rightarrow S$ is not shape operator of a surface in $\mathbb{R}^3$.*

Thus there is no ovaloid in $\mathbb{R}^3$, different from round spheres, with $k_2 = f(k_1)$ and monotone decreasing f ($k_1 = H + \sqrt{H^2 - K}$, $k_2 = H - \sqrt{H^2 - K}$).

Examples: *(a) $k_1 = 2 - v^2, k_2 = 1 + u^2$ is impossible in $\mathbb{R}^3$. Compare also section 7.4 below.*
(b) There is no surface in $\mathbb{R}^3$ such that k_1, k_2 have different constant values $\neq 0$.

Remarks: Proposition 3 is not true for all surfaces of class C^3.
H. Weyl [12], p. 642, derived his identity in order to get a priori bounds for H in terms of the metric. Introducing H, K and the components g_{ij}, b_{ij} of I and II in arbitrary coordinates u^1, u^2, W is given by

$$W = \frac{1}{4(H^2 - K)}[b^{*ij} 2\partial_j H - g^{ij}\partial_j K]\partial_i, \quad b^{*ij} = \varepsilon^{ip}\varepsilon^{jq} b_{pq}$$

with the usual definition of the ε-tensor.

5 k_1, k_2 as functions of curvature line coordinates

For $k_1 \neq k_2$, there are local coordinates u, v such that $(S^i{}_j) = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$. We define

(8)
$$P(u,v) = exp\{-\int_0^v \frac{\partial_v k_1}{k_1 - k_2} dv\}, \quad P(u,0) = 1,$$
$$Q(u,v) = exp\{\int_0^u \frac{\partial_u k_2}{k_1 - k_2} du\}, \quad Q(0,v) = 1.$$

For the required metric we write

(9)
$$ds^2 = Edu^2 + Gdv^2; \quad \sqrt{E} = \varrho, \ \sqrt{G} = \sigma.$$

Passing from e_1, e_2 to the Gauss basis $\partial_u = \varrho e_1$, $\partial_v = \sigma e_2$, one gets $\omega(e_1) = -\varrho_v/(\varrho\sigma)$, $\omega(e_2) = \sigma_u/(\varrho\sigma)$, such that (6) is equivalent to

(10)
$$\frac{\varrho_v}{\varrho} = -\frac{\partial_v k_1}{k_1 - k_2}, \quad \frac{\sigma_u}{\sigma} = \frac{\partial_u k_2}{k_1 - k_2};$$

according to (4).

Proposition 4: *With respect to curvature line coordinates, the metric (9) is determined by k_1, k_2 up to factors $\varphi(u), \psi(v) > 0$:*

$$\varrho(u,v) = \varphi(u)P(u,v), \quad \sigma(u,v) = \psi(v)Q(u,v) \tag{11}$$

with P, Q from (8).

Proof: (10) $\Leftrightarrow$ $(\log \frac{\varrho}{P})_v = 0$ and $(\log \frac{\sigma}{Q})_u = 0$. ∎

$\varphi(u)$ and $\psi(v)$ fix the arc length on the curves $(u, 0)$ and $(0, v)$ with respect to the required metric. (11) implies

Proposition 5: *$k_1 \neq k_2$ are principal curvatures of a surface in $\widetilde{M}(c)$ if and only if there exist functions $\varphi(u), \psi(v) > 0$ such that*

$$-\frac{1}{PQ}[\frac{1}{\psi}(\frac{P_v}{\psi Q})_v + \frac{1}{\varphi}(\frac{Q_u}{\varphi P})_u] = c + k_1 k_2 \tag{12}$$

with P, Q from (8).

Proof: K_I can be calculated from (9) by the formula

$$K_I = -\frac{1}{\varrho\sigma}[(\frac{\varrho_v}{\sigma})_v + (\frac{\sigma_u}{\varrho})_u]. \tag{13}$$

Thus (11) and (12) are equivalent to (3) and (2). ∎

(12) shows that the functions k_1, k_2 in the special coordinates u, v are not independent; "in general" we have case 1 of the introduction. If one tries to prescribe k_1, then there will be strong restrictions on k_2.

6 S-deformability and S-rigidity

Suppose that there is a surface with principal curvatures k_1, k_2; thus (9) and (10) hold and $K_I = c + k_1 k_2$ with K_I from (13). (11) implies that for a second surface with the same k_1, k_2 one has

$$\bar{\varrho}(u,v) = \lambda(u)\varrho(u,v), \qquad \bar{\sigma}(u,v) = \mu(v)\sigma(u,v); \qquad \lambda, \mu > 0.$$

We define

$$\frac{1}{\lambda(u)^2} - 1 = \alpha(u), \quad \frac{1}{\mu(v)^2} - 1 = \beta(v); \quad \alpha, \beta > -1.$$

Proposition 6: *A surface in $\widetilde{M}(c)$ is locally S-deformable if and only if there exist functions $(\alpha(u), \beta(v))$ not identically $(0,0)$ such that*

$$(\frac{\sigma_u}{\varrho})_u \alpha + \frac{1}{2}\frac{\sigma_u}{\varrho}\alpha' + (\frac{\varrho_v}{\sigma})_v \beta + \frac{1}{2}\frac{\varrho_v}{\sigma}\beta' = 0. \tag{14}$$

Proof: For the second surface, (2) is equivalent to an equation for $\overline{K}_I$ analogous to (12), if one replaces P, Q, φ, ψ in (12) by $\varrho, \sigma, \lambda, \mu$. Substraction of (13) from this equation yields (14) if and only if (2) holds for the second surface. ∎

Proposition 7: *If $\alpha(u), \beta(v)$ is a solution of (14), then locally*

$$ds_\varepsilon^2 = \frac{E}{1+\varepsilon\alpha}du^2 + \frac{G}{1+\varepsilon\beta}dv^2, \quad |\varepsilon| < \varepsilon_0$$

is the metric of a one-parameter family of surfaces with the same S.

Proof: (14) is homogeneous in α, β; for small ε one has $\varepsilon\alpha, \varepsilon\beta > -1$. ∎

Remark: In [5] it is shown that the space of solutions (α, β) of (14), for S-deformable surfaces, has dimension 1, 2 or 3.

7 Examples in $\mathbb{R}^3$

7.1 Enneper's minimal surface

$$w = u + iv, \quad x(u,v) = Re\int (1-w^2, i(1+w^2), 2w)dw,$$

$$ds^2 = E(du^2+dv^2), \quad E = (1+u^2+v^2)^2, \quad k_1 = -k_2 = -\frac{2}{E}\,.$$

(14) takes the form

$$(1+v^2-u^2)\alpha + (1+u^2-v^2)\beta + \frac{1}{2}(1+u^2+v^2)(u\alpha' + v\beta') = 0\,. \tag{15}$$

We assume $\alpha(0) = \varepsilon > 0$. For $u = 0$ and $v = 0$ or 1, (15) yields $\beta(0) = -\varepsilon$ and $\beta'(1) = -2\varepsilon$. For $v = 0$ and $v = 1$, from (15) there result two equations for $\alpha(u)$ and $\alpha'(u)$. By elimination of α' and analogous calculations for β, we find: The general solution of (15) is

$$\alpha = \varepsilon(1+u^2), \quad \beta = -\varepsilon(1+v^2); \quad 0 < \varepsilon < 1; \quad \beta > -1 \Leftrightarrow |v| < \sqrt{1/\varepsilon - 1}\,.$$

Conclusion: *The Enneper minimal surface is globally S-rigid, but every strip $|v| < \sqrt{1/\varepsilon - 1}$ is S-deformable.*

7.2 Example of a locally rigid minimal surface

$$ds^2 = E(du^2+dv^2); \quad k_1 = -k_2 = \frac{1}{E}, \quad f = \log E, \quad f_{uu} + f_{vv} = 2e^{-f}.$$

(14) takes the form

$$\alpha f_{uu} + \frac{1}{2}\alpha' f_u + \beta f_{vv} + \frac{1}{2}\beta' f_v = 0. \tag{16}$$

$\triangle f = 2e^{-f}$ has a solution $f(u,v)$ with $f(u,0) = u^4$ and $f_v(u,0) = u$. Taking $v = 0$ in (16) yields

$$12u^2\alpha + 2u^3\alpha' + \beta(0)[2e^{-u^4} - 12u^2] + \frac{1}{2}\beta'(0)u = 0. \tag{17}$$

This implies $\beta(0) = 0$ and, after dividing by u, implies also $\beta'(0) = 0$. This gives $6\alpha + u\alpha' = 0 \Rightarrow u^6\alpha = const \Rightarrow \alpha = 0$. (16) $\Rightarrow (\beta f_v^2)_v = 0 \Rightarrow \beta f_v^2 = \gamma(u)$; $v = 0 \Rightarrow \beta(v) f_v(u,v) = 0$; $f_{vu}(u,0) = 1 \Rightarrow \beta = 0$:

Conclusion: *The surface is S-rigid in a neighbourhood of* $(0,0)$ (Example for case 2).

7.3 S-deformations with constant α, β

$$ds^2 = E(du^2 + dv^2); \quad k_1 = H + \frac{1}{E}, \quad k_2 = H - \frac{1}{E}, \quad H = const;$$

$f = \log E$ is a solution of

$$f_{uu} + f_{vv} = 2(e^{-f} - H^2 e^f). \tag{18}$$

(14) for constant α, β:

$$\alpha f_{uu} + \beta f_{vv} = 0. \tag{19}$$

Choose a non-constant solution $\varphi(t)$ of $\varphi'' = 2(e^{-\varphi} - H^2 e^{\varphi})$ and choose constants

$$a, b \in \mathbb{R}; \quad a, b > 0, \quad a^2 + b^2 = 1, \tag{20}$$

and define

$$f(u,v) = \varphi(au + bv).$$

Since $f_{uu} = a^2\varphi''$ and $f_{vv} = b^2\varphi''$, f is a solution of (18). For

$$\alpha = -b^2, \quad \beta = a^2$$

f also is a solution of (19) and consequently

$$ds_\varepsilon^2 = E\left(\frac{du^2}{1 - \varepsilon b^2} + \frac{dv^2}{1 + \varepsilon a^2}\right), \quad 0 \le \varepsilon \le 1$$

is the metric of a family of H-surfaces, where the correspondence between the surfaces preserves S and multiplies the area element with a constant factor. With the special values

$$a^2 = \frac{\sqrt{5} - 1}{2}, \quad b^2 = \frac{3 - \sqrt{5}}{2}$$

the correspondence between the two surfaces for $\varepsilon = 0$ and $\varepsilon = 1$ preserves the area element. Since the translations $(u,v) \to (u,v) + t(-b,a)$, for every $t \in \mathbb{R}$, preserve k_1, k_2 and ds_ε^2, and since the orbits are different from the curvature lines, our surfaces are helicoidal surfaces.

Conclusion: *The helicoidal surfaces with constant H, i.e. the associated surfaces of the catenoid and of the Delaunay surfaces, allow global S-deformations with constant α, β. There are pairs where the correspondence is area preserving as well.*

7.4 Cyclides of Dupin

These surfaces are characterized by the condition $e_1k_1 = e_2k_2 = 0$, i.e. $k_1 = k_1(v), k_2 = k_2(u)$. This means that both focal surfaces degenerate to curves. We consider the case of regular tori in $\mathbb{R}^3$. They can be parametrized as follows:

Focal curves: $y(v) = (a\cos v, b\sin v, 0), \quad z(u) = \frac{1}{\cos u}(e, 0, b\sin u).$

Principal radii: $r(v) = c+e\cos v, \; s(u) = c+\frac{a}{\cos u}, \; |s| > r, \; a > b > 0, \; e^2 = a^2-b^2, \; e < c < a,$

$$N(u,v) = \frac{1}{s-r}(z-y) \Rightarrow |N| = 1.$$

Surface: $x(u,v) = y(v) - r(v)N(u,v) = z(u) - s(u)N(u,v)$

$$\Rightarrow N \perp x_u, x_v; \quad k_1 = \frac{1}{c+e\cos v}, \quad k_2 = \frac{cosu}{a+c\cos u}.$$

Observe that the denominator $\cos u$ disappears in $x(u,v)$. The range of k_1, k_2 is

$$k_1(v) \in I_1 = [\frac{1}{c+e}, \frac{1}{c-e}]; \quad k_2(v) \in I_2 = [-\frac{1}{a-c}, \frac{1}{a+c}]. \tag{21}$$

Assertion: *Dupin cyclides are S-deformable in the neigbourhood of every point, but globally they are S-rigid.*

Proof: The cyclides depend on the constants (a,c,e).

a) If $(k_1(v_0), k_2(u_0))$ is an interior point of $I_1 \times I_2$, then for all cyclides $(\tilde{a}, \tilde{c}, \tilde{e})$ near (a,c,e) the equations

$$k_1(v) = \tilde{k}_1(\tilde{v}), \quad k_2(u) = \tilde{k}_2(\tilde{u})$$

locally define an S-preserving diffeomorphism.

b) The same is true if I_1 and $\tilde{I}_1$ have the left (or the right) endpoints in common or if the same is true for I_2, $\tilde{I}_2$ (or both), since for endpoints of I_α we have $k'_\alpha = 0$, $k''_\alpha \neq 0$. The endpoints are determined by $x_1 = c+e$, $y_1 = c-e$, $x_2 = a+c$, $y_2 = a-c$ where three constants are free, but $x_1 + y_1 = x_2 - y_2$. If one or two of the endpoints are fixed, there are cyclides $(\tilde{a}, \tilde{c}, \tilde{e})$ near (a,c,e) (depending on 2 or 1 parameter) which locally have the same S. Thus the surface is locally S-deformable near every point. Notice that a surface with the same S as a cyclide is itself a cyclide. If two cyclides have globally the same S, then $I_1 \times I_2 = \tilde{I}_1 \times \tilde{I}_2 \Rightarrow (\tilde{a}, \tilde{c}, \tilde{e}) = (a,c,e)$. ∎

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